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Advanced Mathematical Methods and Software Platform for Solving Multi-Physics, Multi-Domain Problems on Modern Computer Architectures: Applications to Environmental Engineering and Medical Problems

Work Package 2 - Task 2.2
Interface Relaxation Methods (IR)

Kick Off Meeting - 2012/07/21 – Chania, Crete, Greece

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outline

Timetables - Deliverables

Scientific Description

Timetables - Deliverables

					Start Date:			01/01/02012			End Date:			31/09/2015							
				Month	2012				2013				2014				2015				
					jan-mar	apr-jun	jul - sep	oct - dec	jan-mar	apr-jun	jul - sep	oct - dec	jan-mar	apr-jun	jul - sep	oct - dec	jan-mar	apr-jun	jul	aug	sep
		WP Title:	Research Team:	List of delivarables:	1-3	4-6	7-9	9-12	13-15	16-18	19-21	22-24	25-27	28-30	31-33	34-36	37-39	40-42	43	44	45
1	WP 1	COORDINATION & PLANNING	1 (2, 3)	3 (ANNUAL)INTERMEDIATE TECHNICAL REPORTS																	
			1 (2, 3)	1 FINAL TECHNICAL REPORT																	
2	WP 2	NUMERICAL & ANALYTICAL METHODS FOR DISCONTINUOUS MULTIDOMAIN PROBLEMS (DISCONTINOUS COLLOCATION METHODS, RELAXATION METHODS, STOCHASTIC-DETERMINISTIC HYBRID METHODS, FOKAS METHODS)	1	2.1 DISCONTINUOUS COLLCATION: 1 TECHNICAL REPORT / 2 PAPERS / SOFTWARE CODE																	
			2 (1)	2.2 RELAXATION METHODS: 1 TECHNICAL REPORT / 2 PAPERS / SOFTWARE	←		Start								End			→			
			2	2.3 STOCHASTIC-DETERMINISTIC HYBRID METHODS: 1 TECHNICAL REPORT / 2 PAPERS / SOFTWARE																	
			1 (3)	2.4 FOKAS METHODS: 1 TECHNICAL REPORT / 2 PAPERS																	
3	WP 3	MAPPING & IMPLEMENTATION ON STATE-OF-THE-ART COMPUTING ENVIROMENTS (CLUSTERS-GRIDS-CLOUDS-FPGAS-RECONFIGURABLE ARRAYS)	3 (1)	3.1 CLUSTERS-GRIDS-CLOUDS: 1 TECHNICAL REPORT / 3 PAPERS / SOFTWARE																	
			2	3.2 FPGAS-RECONFIGURABLE ARRAYS: 1 TECHNICAL REPORT / 2 PAPERS / SOFTWARE																	
4	WP 4	INTEGRATING, EVALUATING AND VALIDATING THE RESULTS ON ENVIRONMENTAL AND MEDICAL PROBLEMS	2, 3 (1)	4.1 INTEGRATION: 1 TECHNICAL REPORT / SOFTWARE																	
			1 (2, 3)	4.2 MEDICAL PROBLEMS: 1 TECHNICAL REPORT / 1 PAPERS																	
			1 (2, 3)	4.3 ENVIROMENTAL PROBLEMS: 1 TECHNICAL REPORT / 1 PAPERS																	
5	WP 5	PROJECT EVALUATION	1	PROJECT EVALUATION BY EXTERNAL EVALUATORS																	
6	WP 6	DISSEMINATION OF THE RESULTS	1 (2, 3)	WEB SITE, WORKSHOP: WORKSHOP PROCEEDINGS																	

Timetable - Deliverables

- Timetable
  - Start : 1/1/2012
  - End : 30/6/2015
  - Duration : 40 months
- Deliverables
  - 1 technical report (τεχνικές εκθέσεις)
  - 3 research papers (επιστημονικά άρθρα)

Scientific Description

Interface Relaxation Methods

Interface Relaxation (IR) Methods for General Elliptic PDEs

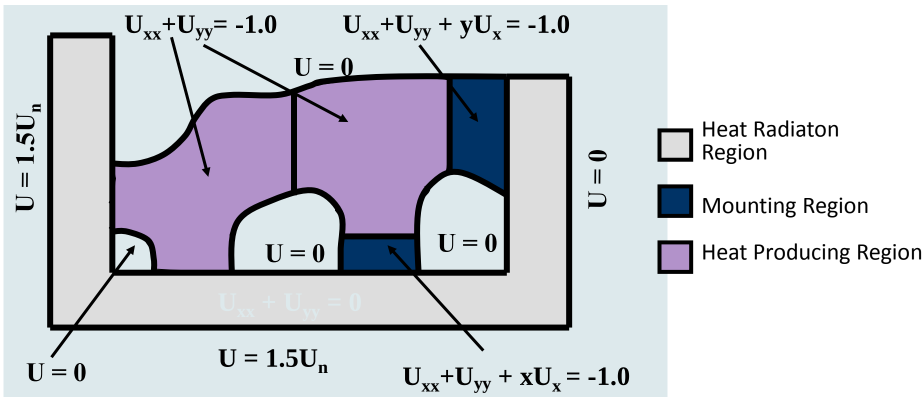
1. Introduction
   - Basic target.
   - A composed Problem.
   - Basic Methodology
   - General stuff.
2. IR Methods.
   - *AVE*.
   - *GEO*.
   - *ROB*.
3. Convergence Analysis.
4. Numerical Results.

Main Target

The creation of a general methodology for the solution of the composite PDE problems with the following properties:

1. *Consists of a set of collaboration solvers for local (simple) problems.*
2. *Allow the choice of the most appropriate discretization method of the local problem.*
3. *Simplifies the geometry and the physics of the global problem.*
4. *Software reuse.*
5. *Wide applicability and high efficiency.*
6. *Increased adaptivity and inherent parallelism.*

A Composed PDE Problem



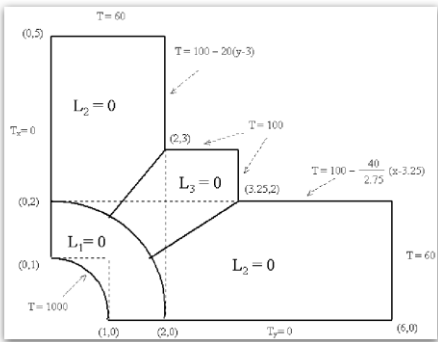
Basic Methodology

1. Split the problem in the PDE level.
2. Use solution's properties to set conditions on the common boundaries (interfaces)
3. Give initial values on the interfaces.
4. Solve each PDE locally.
5. Use relaxation method to smooth the solution/derivatives or/and operators on the interfaces.
6. Check convergence criteria and go back to 4.

Basic Methodology

Initial global composed PDE problem

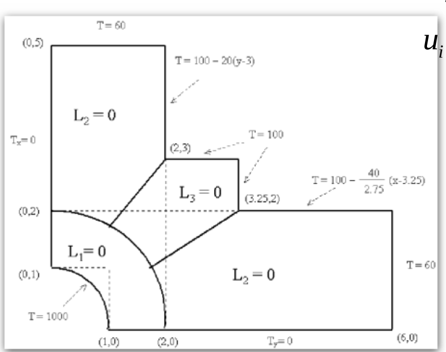
$$Lu = f \ , \ \text{ in } \ \Omega \setminus \partial\Omega$$
$$u = u^b \ , \ \text{ on } \ \partial\Omega$$

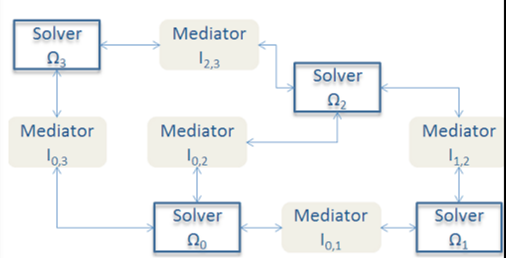


Basic Methodology

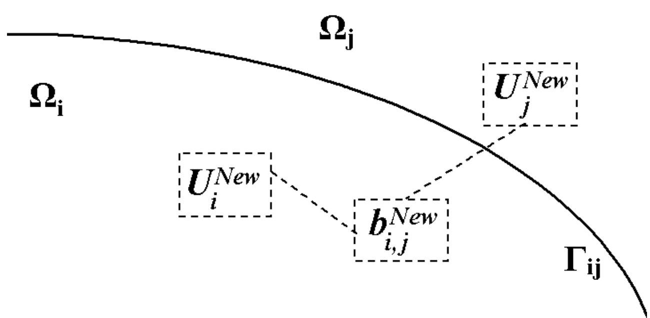
Set of simple PDE subproblems

$$L_i u_i = f_i \ , \ \text{ in } \ \Omega_i$$
$$G_{ij} u_i = 0 \ , \ \text{ on } \ (\partial\Omega_i \cap \partial\Omega) \setminus \partial\Omega \quad \forall i \neq j$$
$$u_i = u_i^b \ , \ \text{ on } \ \partial\Omega_i \cap \partial\Omega$$





Basic Methodology



Relaxation:

$$G_{ij}\left(U_i^{New}, U_j^{New}, \frac{\partial U_i^{New}}{\partial \eta}, \frac{\partial U_j^{New}}{\partial \eta} \right) = 0$$

Overview in IR Methods

- Primitive Relaxers:
  - About 10 proposed/implemented/analysed.
  - Smooth values and normal derivatives in various ways
- Advanced Relaxers:
  - Smooth additional operators (continuity of mass, temperature, conservation of energy/momentum, equilibrium conditions, Lagrange multipliers, Steklov-Poincare operators, etc.)
- Differences in convergence and applicability.

AVE

For k=0,1,2,…

- Define:

$$g_i = \beta_i \left.\frac{du_i^{(2k)}}{dx}\right|_{x=x_i} + (1-\beta_i)\left.\frac{du_{i+1}^{(2k)}}{dx}\right|_{x=x_i}, i=1,\ldots,p-1$$

- Solve the Neumann problems:

$$L_1 u_1^{(2k+1)} = f_1, \text{ στο } \Omega_1 \qquad u_1^{(2k+1)}\Big|_{x=x_0} = 0 \qquad \left.\frac{du_1^{(2k+1)}}{dx}\right|_{x=x_1} = g_1$$

$$L_i u_i^{(2k+1)} = f_i, \text{ στο } \Omega_i \qquad \left.\frac{du_i^{(2k+1)}}{dx}\right|_{x=x_{i-1}} = g_{i-1} \qquad \left.\frac{du_i^{(2k+1)}}{dx}\right|_{x=x_i} = g_i$$

$i=2,\ldots,p-1$

$$L_p u_p^{(2k+1)} = f_p, \text{ στο } \Omega_p \qquad \left.\frac{du_p^{(2k+1)}}{dx}\right|_{x=x_{p-1}} = g_{p-1} \qquad u_p^{(2k+1)}\Big|_{x=x_p} = 0$$

AVE (cont'd)

- Define:

$$h_i = \alpha_i u_i^{(2k+1)}\Big|_{x=x_i} + (1-\alpha_i) u_{i+1}^{(2k+1)}\Big|_{x=x_i}, i=1,\ldots,p-1$$

- Solve the Dirichlet problems:

$$L_1 u_1^{(2k+2)} = f_1, \text{ στο } \Omega_1 \qquad u_1^{(2k+2)}\Big|_{x=x_0} = 0 \qquad u_1^{(2k+2)}\Big|_{x=x_1} = h_1$$

$$L_i u_i^{(2k+2)} = f_i, \text{ στο } \Omega_i \qquad u_i^{(2k+2)}\Big|_{x=x_{i-1}} = h_{i-1} \qquad u_i^{(2k+2)}\Big|_{x=x_i} = h_i$$

$i=2,\ldots,p-1$

$$L_p u_p^{(2k+2)} = f_p, \text{ στο } \Omega_p \qquad u_p^{(2k+2)}\Big|_{x=x_{p-1}} = h_{p-1} \qquad u_p^{(2k+2)}\Big|_{x=x_p} = 0$$

GEO

For k=0,1,2,...

- Define: $$g_i = \left.\frac{u_i^{(k)} + u_{i+1}^{(k)}}{2}\right|_{x=x_i} + \rho_i \left.\left(-\frac{du_i^{(k)}}{dx} + \frac{du_{i+1}^{(k)}}{dx}\right)\right|_{x=x_i}, i = 1,\dots,p-1$$

- Solve the Dirichlet problems:

$$L_1 u_1^{(k+1)} = f_1, \text{ στο } \Omega_1 \qquad u_1^{(k+1)}\Big|_{x=x_0} = 0 \qquad u_1^{(k+1)}\Big|_{x=x_1} = g_1$$

$$L_i u_i^{(k+1)} = f_i, \text{ στο } \Omega_i \qquad u_i^{(k+1)}\Big|_{x=x_{i-1}} = g_{i-1} \qquad u_i^{(k+1)}\Big|_{x=x_i} = g_i$$

$i=2,\dots,p-1$

$$L_p u_p^{(k+1)} = f_p, \text{ στο } \Omega_p \qquad u_p^{(k+1)}\Big|_{x=x_{p-1}} = g_{p-1} \qquad u_p^{(k+1)}\Big|_{x=x_p} = 0$$

ROB

For k=0,1,2,...

- Define:

$$\left.\begin{aligned} g_i^i &= \left.\frac{du_{i+1}^{(k)}}{dx}\right|_{x=x_i} + \lambda_i u_{i+1}^{(k)}\Big|_{x=x_i} \\ g_i^{i+1} &= -\left.\frac{du_i^{(k)}}{dx}\right|_{x=x_i} + \lambda_i u_i^{(k)}\Big|_{x=x_i} \end{aligned}\right\} i = 1,\dots,p-1$$

- Solve the problems:

$$L_1 u_1^{(k+1)} = f_1, \text{ στο } \Omega_1 \qquad u_1^{(k+1)}\Big|_{x=x_0} = 0 \qquad \left.\frac{du_1^{(k+1)}}{dx}\right|_{x=x_1} + \lambda_1 u_1^{(k+1)}\Big|_{x=x_1} = g_1^1$$

$$L_i u_i^{(k+1)} = f_i, \text{ στο } \Omega_i \qquad -\left.\frac{du_i^{(k+1)}}{dx}\right|_{x=x_{i-1}} + \lambda_{i-1} u_i^{(k+1)}\Big|_{x=x_{i-1}} = g_{i-1}^i \qquad \left.\frac{du_i^{(k+1)}}{dx}\right|_{x=x_i} + \lambda_i u_i^{(k+1)}\Big|_{x=x_i} = g_i^i$$

$i=2,\dots,p-1$

$$L_p u_p^{(k+1)} = f_p, \text{ στο } \Omega_p \qquad -\left.\frac{du_p^{(k+1)}}{dx}\right|_{x=x_{p-1}} + \lambda_{p-1} u_p^{(k+1)}\Big|_{x=x_{p-1}} = g_{p-1}^p \qquad u_p^{(k+1)}\Big|_{x=x_p} = 0$$

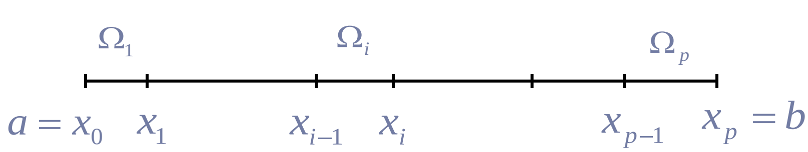
Convergence Analysis

- The model problem.
- Convergence conditions and "optimum" values for the relaxation parameters of AVE.
- Optimum values for the relaxation parameters of ROB.
- Convergence conditions for GEO.

Model problem (1-dim)

$$-u_{xx} + \gamma^2 u = f, \quad x \in \Omega \equiv (a,b)$$

$$u(a) = u(b) = 0$$



It is known that:

The solution of $-u_{xx} + \gamma^2 u = 0\,, \quad x \in (a,b)$

with
$$c_1 u'(a) + c_2 u(a) = \upsilon_1\,,$$
$$c_3 u'(b) + c_4 u(b) = \upsilon_2$$

is given by

$$\frac{-(c_3\gamma + c_4)e^{\gamma(b-x)} + (-c_3\gamma + c_4)e^{-\gamma(b-x)}}{-(-c_1\gamma + c_2)(c_3\gamma + c_4)e^{\gamma(b-a)} + (c_1\gamma + c_2)(-c_3\gamma + c_4)e^{-\gamma(b-a)}}\upsilon_1 +$$
$$\frac{-(-c_1\gamma + c_2)e^{\gamma(x-a)} + (c_1\gamma + c_2)e^{-\gamma(x-a)}}{-(-c_1\gamma + c_2)(c_3\gamma + c_4)e^{\gamma(b-a)} + (c_1\gamma + c_2)(-c_3\gamma + c_4)e^{-\gamma(b-a)}}\upsilon_2.$$

D.E. for the error functions *(AVE)*

- $\varepsilon_i^{(k)} = u_i^{(k)} - u_i$, $i=1,\ldots,p$, $k=0,1,\ldots$
- Neumann steps (similar for the Dirichlet steps):

$$g\varepsilon_i = \beta_i \left.\frac{d\varepsilon_i^{(2k)}}{dx}\right|_{x=x_i} + (1-\beta_i)\left.\frac{d\varepsilon_{i+1}^{(2k)}}{dx}\right|_{x=x_i}, i = 1,\ldots, p-1$$

$$L_1\varepsilon_1^{(2k+1)} = 0, \text{ in } \Omega_1 \qquad \left.\varepsilon_1^{(2k+1)}\right|_{x=x_0} = 0 \qquad \left.\frac{d\varepsilon_1^{(2k+1)}}{dx}\right|_{x=x_1} = g\varepsilon_1$$

$$L_i\varepsilon_i^{(2k+1)} = 0, \text{ in } \Omega_i \qquad \left.\frac{d\varepsilon_i^{(2k+1)}}{dx}\right|_{x=x_{i-1}} = g\varepsilon_{i-1} \qquad \left.\frac{d\varepsilon_i^{(2k+1)}}{dx}\right|_{x=x_i} = g\varepsilon_i \qquad i=2,\ldots,p\text{-}1$$

$$L_p\varepsilon_p^{(2k+1)} = 0, \text{ in } \Omega_p \qquad \left.\frac{d\varepsilon_p^{(2k+1)}}{dx}\right|_{x=x_{p-1}} = g\varepsilon_{p-1} \qquad \left.\varepsilon_p^{(2k+1)}\right|_{x=x_p} = 0$$

Error functions (*AVE)*

Neumann:

$$\varepsilon_i^{(2k+1)}(x) = \frac{-\cosh(\gamma_i(x_i - x))}{\gamma_i \sinh(\gamma_i \ell_i)}\left(\beta_{i-1} d\varepsilon_{i-1,i-1}^{(2k)} + (1-\beta_{i-1}) d\varepsilon_{i,i-1}^{(2k)}\right) + \frac{\cosh(\gamma_i(x - x_{i-1}))}{\gamma_i \sinh(\gamma_i \ell_i)}\left(\beta_i d\varepsilon_{i,i}^{(2k)} + (1-\beta_i) d\varepsilon_{i+1,i}^{(2k)}\right), \quad i = 2,\ldots,p-1$$

Dirichlet:

$$\varepsilon_i^{(2k+2)}(x) = \frac{\sinh(\gamma_i(x_i - x))}{\sinh(\gamma_i \ell_i)}\left(\alpha_{i-1} \varepsilon_{i-1,i-1}^{(2k+1)} + (1-\alpha_{i-1}) \varepsilon_{i,i-1}^{(2k+1)}\right) + \frac{\sinh(\gamma_i(x - x_{i-1}))}{\sinh(\gamma_i \ell_i)}\left(\alpha_i \varepsilon_{i,i}^{(2k+1)} + (1-\alpha_i) \varepsilon_{i+1,i}^{(2k+1)}\right), \quad i = 2,\ldots,p-1$$

where $d\varepsilon_{i,j}^{(2k)} = \dfrac{d\varepsilon_i^{(2k)}(x_j)}{dx}, \quad \varepsilon_{i,j}^{(2k+1)} = \varepsilon_i^{(2k+1)}(x_j).$

Relation between two consecutive errors (*AVE)*

- $\underline{\varepsilon}^{(k)} = \left[\varepsilon_{1,1}^{(k)}, \varepsilon_{2,2}^{(k)}, \ldots, \varepsilon_{p-1,p-1}^{(k)}\right]^T, \quad \underline{d\varepsilon}^{(k)} = \left[d\varepsilon_{1,1}^{(k)}, d\varepsilon_{2,2}^{(k)}, \ldots, d\varepsilon_{p-1,p-1}^{(k)}\right]^T,$
- $\underline{\varepsilon}^{(2k+2)} = M^D \underline{d\varepsilon}^{(2k+1)}, \quad \underline{d\varepsilon}^{(2k+1)} = M^N \underline{\varepsilon}^{(2k)}$
- $M_{i+1,i}^D = -\dfrac{2\alpha_i}{m_i \gamma_i}, \quad i = 1,\ldots,p-2, \quad M_{i,i+1}^D = \dfrac{2(1-\alpha_i)}{m_{i+1}\gamma_{i+1}}, \quad i = 2,\ldots,p-1,$

  $$M_{i,i}^D = \frac{\alpha_i n_i}{m_i \gamma_i} - \frac{(1-\alpha_i) n_{i+1}}{m_{i+1}\gamma_{i+1}}, \quad i = 2,\ldots,p-2$$
- $M_{i+1,i}^N = -\dfrac{2\beta_i \gamma_i}{m_i}, \quad i = 1,\ldots,p-2, \quad M_{i,i+1}^N = \dfrac{2(1-\beta_i)\gamma_{i+1}}{m_{i+1}}, \quad i = 2,\ldots,p-1,$

  $$M_{i,i}^N = \frac{\beta_i n_i \gamma_i}{m_i} - \frac{(1-\beta_i) n_{i+1}\gamma_{i+1}}{m_{i+1}}, \quad i = 2,\ldots,p-2$$

Convergence, optimum values *(AVE)*

Theorem: Let $\Omega = \bigcup_{i=1}^{p} \overline{\Omega_i}, \quad \Omega_i = (x_{i-1}, x_i)$ of length $\ell_i, \; i=1,\ldots,p$ and $\gamma_i=\gamma$, $i=1,\ldots,p$. If $\ell_i > \frac{\ln(1+\sqrt{2})}{\gamma}, \; i=1,\ldots,p,$ then $\|M^N M^D\|_\infty$ is minimized by

$$\alpha_i = \frac{m_i n_{i+1}}{m_i n_{i+1} + m_{i+1} n_i} \quad \text{and} \quad \beta_i = \frac{m_i n_{i+1}}{m_i n_{i+1} + m_{i+1} n_i}, \; i=2,\ldots,p\text{-}2.$$

D.E. for the error functions *(GEO)*

$$g\varepsilon_i = \left.\frac{\varepsilon_i^{(k)} + \varepsilon_{i+1}^{(k)}}{2}\right|_{x=x_i} + \rho_i \left.\left(-\frac{d\varepsilon_i^{(k)}}{dx} + \frac{d\varepsilon_{i+1}^{(k)}}{dx}\right)\right|_{x=x_i}, i=1,\ldots,p-1$$

$$L_1\varepsilon_1^{(k+1)} = 0, \quad \text{στο } \Omega_1$$

$$\left.\varepsilon_1^{(k+1)}\right|_{x=x_0} = 0$$

$$\left.\varepsilon_1^{(k+1)}\right|_{x=x_1} = g\varepsilon_1$$

$$L_i\varepsilon_i^{(k+1)} = 0, \quad \text{στο } \Omega_i$$

$$\left.\varepsilon_i^{(k+1)}\right|_{x=x_{i-1}} = g\varepsilon_{i-1}$$

$$\left.\varepsilon_i^{(k+1)}\right|_{x=x_i} = g\varepsilon_i$$

$i=2,\ldots,p\text{-}1$

$$L_p\varepsilon_p^{(k+1)} = 0, \quad \text{στο } \Omega_p$$

$$\left.\varepsilon_p^{(k+1)}\right|_{x=x_{p-1}} = g\varepsilon_{p-1}$$

$$\left.\varepsilon_p^{(k+1)}\right|_{x=x_p} = 0$$

Error functions *(GEO)*

$$\varepsilon_i^{(k+1)}(x) = \frac{-e^{\gamma_i(x_i-x)} + e^{-\gamma_i(x_i-x)}}{e^{\gamma_i\ell_i} - e^{-\gamma_i\ell_i}}\left(\frac{\varepsilon_{i-1,i-1}^{(k)} + \varepsilon_{i,i-1}^{(k)}}{2} + \rho_{i-1}\left(-d\varepsilon_{i-1,i-1}^{(k)} + d\varepsilon_{i,i-1}^{(k)}\right)\right) + \frac{-e^{\gamma_i(x-x_{i-1})} + e^{-\gamma_i(x-x_{i-1})}}{e^{\gamma_i\ell_i} - e^{-\gamma_i\ell_i}}\left(\frac{\varepsilon_{i,i}^{(k)} + \varepsilon_{i+1,i}^{(k)}}{2} + \rho_i\left(-d\varepsilon_{i,i}^{(k)} + d\varepsilon_{i+1,i}^{(k)}\right)\right), \quad i = 2,\ldots,p-1$$

p=3, $\quad \underline{\varepsilon}^{(k)} = \left[\varepsilon_{1,1}^{(k)}, \varepsilon_{2,1}^{(k)}, \varepsilon_{2,2}^{(k)}, \varepsilon_{3,2}^{(k)}, d\varepsilon_{1,1}^{(k)}, d\varepsilon_{2,1}^{(k)}, d\varepsilon_{2,2}^{(k)}, d\varepsilon_{3,2}^{(k)}\right]^T$

$$\underline{\varepsilon}^{(k+1)} = M\underline{\varepsilon}^{(k)}, \quad k = 0,1,\ldots, \qquad M \in R^{8\times 8},$$

$$A_i = \frac{1}{\tanh(\gamma_i \ell_i)}, \qquad B_i = \frac{1}{\sinh(\gamma_i \ell_i)}, \quad i = 1,2$$

Structure of iteration matrix *(GEO)*

$$M = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\rho_1 & \rho_1 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\rho_1 & \rho_1 & 0 & 0 \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & -\rho_2 & \rho_2 \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & -\rho_2 & \rho_2 \\
\frac{A_1\gamma_1}{2} & \frac{A_1\gamma_1}{2} & 0 & 0 & -\rho_1 A_1\gamma_1 & \rho_1 A_1\gamma_1 & 0 & 0 \\
\frac{-A_2\gamma_2}{2} & \frac{-A_2\gamma_2}{2} & \frac{B_2\gamma_2}{2} & \frac{B_2\gamma_2}{2} & \rho_1 A_2\gamma_2 & -\rho_1 A_2\gamma_2 & -\rho_2 B_2\gamma_2 & \rho_2 B_2\gamma_2 \\
\frac{-B_2\gamma_2}{2} & \frac{-B_2\gamma_2}{2} & \frac{A_2\gamma_2}{2} & \frac{A_2\gamma_2}{2} & \rho_1 B_2\gamma_2 & -\rho_1 B_2\gamma_2 & -\rho_2 A_2\gamma_2 & \rho_2 A_2\gamma_2 \\
0 & 0 & \frac{-A_3\gamma_3}{2} & \frac{-A_3\gamma_3}{2} & 0 & 0 & \rho_2 A_3\gamma_3 & -\rho_2 A_3\gamma_3
\end{bmatrix}$$

Region of convergence (*GEO)*

Lemma: The non-identically zero eigenvalues of Matrix M, are equal to the non-identically zero eigenvalues of $\widetilde{M}$, where

$$\widetilde{M}=\begin{bmatrix}1-\rho_1(\gamma_1 A_1+\gamma_2 A_2) & \rho_2\gamma_2 B_2\\ \rho_1\gamma_2 B_2 & 1-\rho_2(\gamma_2 A_2+\gamma_3 A_3)\end{bmatrix}.$$

Theorem: If $\Omega=\overline{\Omega_1}\bigcup\overline{\Omega_2}\bigcup\overline{\Omega_3},\quad \Omega_i=(x_{i-1},x_i)$ of length $\ell_i,\ i=1,2,3$, GEO converges to the solution of the original problem iff

$$0<\rho_1<\frac{2}{C_1},\quad 0<\rho_2<2\frac{2-\rho_1 C_1}{2C_2-\rho_1(C_1C_2-\gamma_2^2B_2^2)}$$

where $C_i=\gamma_i A_i+\gamma_{i+1}A_{i+1},\quad i=1,2.$

D.E. for the error functions (*ROB)*

$$\left.\begin{aligned} g\varepsilon_i^{i} &= \left.\frac{d\varepsilon_{i+1}^{(k)}}{dx}\right|_{x=x_i} + \lambda_i\,\varepsilon_{i+1}^{(k)}\Big|_{x=x_i}\\ g\varepsilon_i^{i+1} &= -\left.\frac{d\varepsilon_i^{(k)}}{dx}\right|_{x=x_i} + \lambda_i\,\varepsilon_i^{(k)}\Big|_{x=x_i}\end{aligned}\right\} i=1,\ldots,p-1$$

$$L_1\varepsilon_1^{(k+1)}=0,\quad \text{in }\Omega_1$$
$$\varepsilon_1^{(k+1)}\Big|_{x=x_0}=0$$
$$\left.\frac{d\varepsilon_1^{(k+1)}}{dx}\right|_{x=x_1}+\lambda_1\varepsilon_1^{(k+1)}\Big|_{x=x_1}=g\varepsilon_1^1$$

$$L_i\varepsilon_i^{(k+1)}=0,\quad \text{in }\Omega_i$$
$$-\left.\frac{d\varepsilon_i^{(k+1)}}{dx}\right|_{x=x_{i-1}}+\lambda_{i-1}\varepsilon_i^{(k+1)}\Big|_{x=x_{i-1}}=g\varepsilon_{i-1}^i$$
$$\left.\frac{d\varepsilon_i^{(k+1)}}{dx}\right|_{x=x_i}+\lambda_i\varepsilon_i^{(k+1)}\Big|_{x=x_i}=g\varepsilon_i^i$$

$i=2,\ldots,p-1$

$$L_p\varepsilon_p^{(k+1)}=0,\quad \text{in }\Omega_p$$
$$-\left.\frac{d\varepsilon_p^{(k+1)}}{dx}\right|_{x=x_{p-1}}+\lambda_{p-1}\varepsilon_p^{(k+1)}\Big|_{x=x_{p-1}}=g\varepsilon_{p-1}^p$$
$$\varepsilon_p^{(k+1)}\Big|_{x=x_p}=0$$

Error functions (*ROB*)

- $$\varepsilon_i^{(k+1)}(x) = \frac{-(\gamma_i+\lambda_i)e^{\gamma_i(x_i-x)} + (-\gamma_i+\lambda_i)e^{-\gamma_i(x_i-x)}}{(-\gamma_i+\lambda_{i-1})(-\gamma_i+\lambda_i)e^{-\gamma_i\ell_i} - (\gamma_i+\lambda_i)(\gamma_i+\lambda_{i-1})e^{\gamma_i\ell_i}}\left(-d\varepsilon_{i-1,i-1}^{(k)} + \lambda_{i-1}\varepsilon_{i-1,i-1}^{(k)}\right) + \frac{-(\gamma_i+\lambda_{i-1})e^{\gamma_i(x-x_{i-1})} + (-\gamma_i+\lambda_{i-1})e^{-\gamma_i(x-x_{i-1})}}{(-\gamma_i+\lambda_{i-1})(-\gamma_i+\lambda_i)e^{-\gamma_i\ell_i} - (\gamma_i+\lambda_i)(\gamma_i+\lambda_{i-1})e^{\gamma_i\ell_i}}\left(d\varepsilon_{i+1,i}^{(k)} + \lambda_i\varepsilon_{i+1,i}^{(k)}\right), \quad i = 2,\ldots,p-1$$
- $$\underline{\varepsilon}^{(k)} = \left[d\varepsilon_{1,1}^{(k)}, \varepsilon_{1,1}^{(k)}, \varepsilon_{2,1}^{(k)}, d\varepsilon_{2,1}^{(k)}, \ldots, d\varepsilon_{p-1,p-1}^{(k)}, \varepsilon_{p-1,p-1}^{(k)}, \varepsilon_{p,p-1}^{(k)}, d\varepsilon_{p,p-1}^{(k)}\right]^T$$
- $$\underline{\varepsilon}^{(k+1)} = M\underline{\varepsilon}^{(k)}, \quad k = 0,1,\ldots,$$

$$M_{2(i-1),2(i-1)-1} = \frac{1}{d_i}\begin{bmatrix} -(\gamma_i n_i + \lambda_i m_i) & \lambda_{i-1}(\gamma_i n_i + \lambda_i m_i) \\ \gamma_i(\gamma_i m_i + \lambda_i n_i) & -\gamma_i\lambda_{i-1}(\gamma_i m_i + \lambda_i n_i) \end{bmatrix},$$

Structure of iteration matrix (*ROB*)

$$M = \begin{bmatrix}
0 & M_{1,2} & 0 & 0 & 0 & 0 & \cdots & 0 \\
M_{2,1} & 0 & 0 & M_{2,4} & 0 & 0 & \cdots & 0 \\
M_{3,1} & 0 & 0 & M_{3,4} & 0 & 0 & \cdots & 0 \\
0 & 0 & M_{4,3} & 0 & 0 & M_{4,6} & \cdots & 0 \\
0 & 0 & M_{5,3} & 0 & 0 & M_{5,6} & \cdots & 0 \\
\vdots & \vdots & & \ddots & \ddots & \ddots & & \vdots \\
0 & 0 & \cdots & 0 & M_{2(p-1)-2,2(p-1)-3} & 0 & 0 & M_{2(p-1)-2,2(p-1)} \\
0 & 0 & \cdots & 0 & M_{2(p-1)-1,2(p-1)-3} & 0 & 0 & M_{2(p-1)-1,2(p-1)} \\
0 & 0 & \cdots & 0 & 0 & 0 & M_{2(p-1),2(p-1)-1} & 0
\end{bmatrix}$$

Optimal parameters *(ROB)*

Lemma: Let *M* the iteration matrix of the method, then

$\sigma(M)\setminus\{0\} = \sigma(\tilde{M})\setminus\{0\},$ where $\tilde{M}$ has the same structure with *M* and

$$\tilde{M}_{2(i-1),2(i-1)-1} = \frac{\lambda_{i-1}(\gamma_i n_i + \lambda_i m_i) - \lambda_{i-1}(\gamma_i n_i + \lambda_i m_i)}{d_i}, \quad i = 2,\ldots,p-1,$$

$$d_i = (\gamma_i^2 + \lambda_i \lambda_{i-1}) m_i + \gamma_i (\lambda_i + \lambda_{i-1}) n_i.$$

Theorem: Consider the decomposition of *Ω* into *p* subdomains Ω_i of length $\ell_i,\ i=1,\ldots,p$

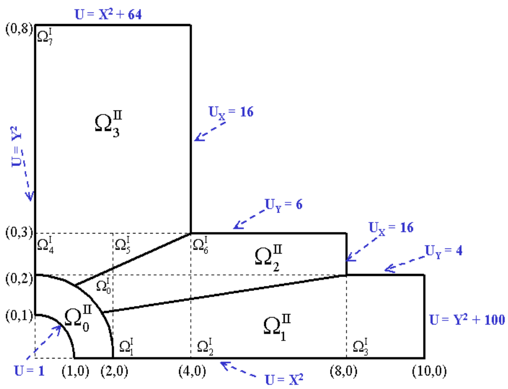
The spectral radious of the iterative matrix is zero if the parameters λ_i , are selected as

$$\lambda_{p-1} = \frac{\gamma_p n_p}{m_p}, \qquad \lambda_{i-1} = \frac{\gamma_i(\gamma_i m_i + \lambda_i n_i)}{\gamma_i n_i + \lambda_i m_i}, \ i = p-1,\ldots,2.$$

Numerical experiments

- 4 composed problems
- History of convergence
- Verification of "optimum" parameters
- Effect of descritization of the domain
- Effect of descritization of the operator

4 composed problems



PDE1: Helmholtz equations with Cartesian decomposition of domain

$$-\Delta u + \gamma_i^2 u = f_i,\ in\ \Omega_i^I, i = 0,..,7$$

$$\gamma_0^2 = \frac{1}{2}\exp^{x+y},\ \gamma_1^2 = 10,\ \gamma_2^2 = 16,\ \gamma_3^2 = 20,$$

$$\gamma_4^2 = 2(\sin((x+y)\pi) + 4),\ \gamma_5^2 = 15,$$

$$\gamma_6^2 = 25,\ \gamma_7^2 = 5\exp(\frac{x+y}{8})$$

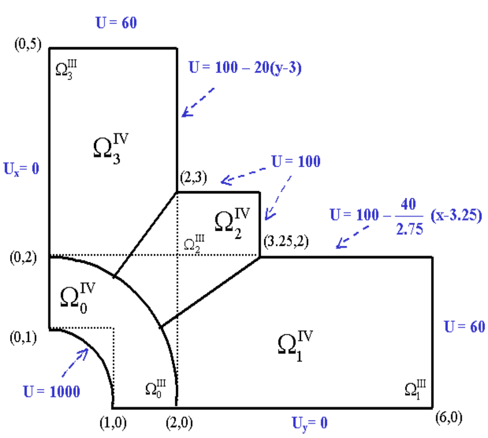
PDE2: Helmholtz equations with general decomposition of domain

$$-\Delta u + \gamma_i^2 u = f_i,\ in\ \Omega_i^{II}, i = 0,..,3$$

$$\gamma_0^2 = \frac{1}{2}\exp^{x+y},\ \gamma_1^2 = 16,\ \gamma_2^2 = 25,$$

$$\gamma_4^2 = \sin((x+y)\pi) + \frac{5}{2}\exp(\frac{x+y}{8}) + 4$$

4 composed problems



PDE3: Linear operator with general decomposition of domain

$$\Delta u + 0.2u + 60(x^2 + y^2 + 2) = 0,\quad in\ \Omega_0^{IV}$$

$$\Delta u + 0.4u = 0,\ in\ \Omega_1^{IV}, \Omega_3^{IV}$$

$$\Delta u - 10(\tfrac{\partial u}{\partial x} + \tfrac{\partial u}{\partial y}) + 0.3u = 0,\ in\ \Omega_2^{IV}$$

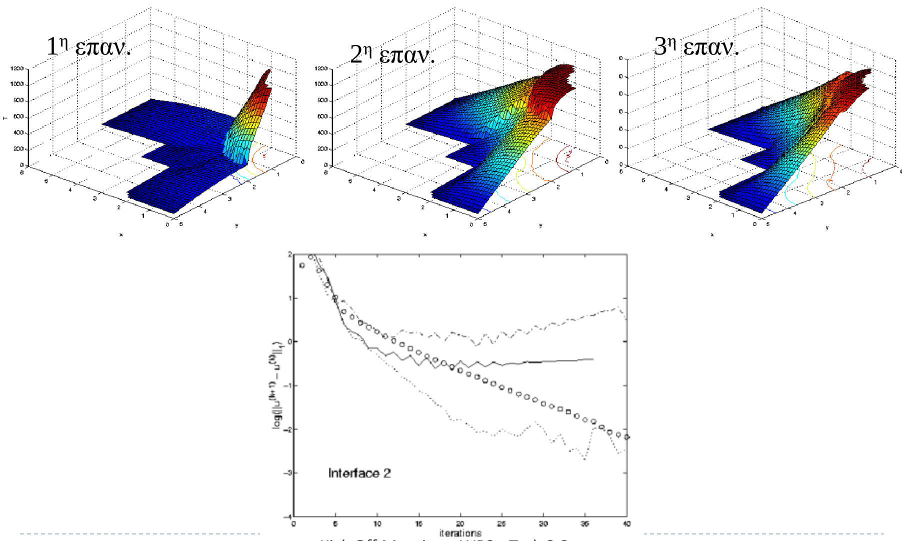
PDE4: Non-Linear operator with general decomposition of domain

$$\Delta u + 0.2u(1 + \tfrac{u}{1000}) + 60(x^2 + y^2 + 2) = 0,\quad in\ \Omega_0^{IV}$$

$$\Delta u + 0.4u = 0,\ in\ \Omega_1^{IV}, \Omega_3^{IV}$$

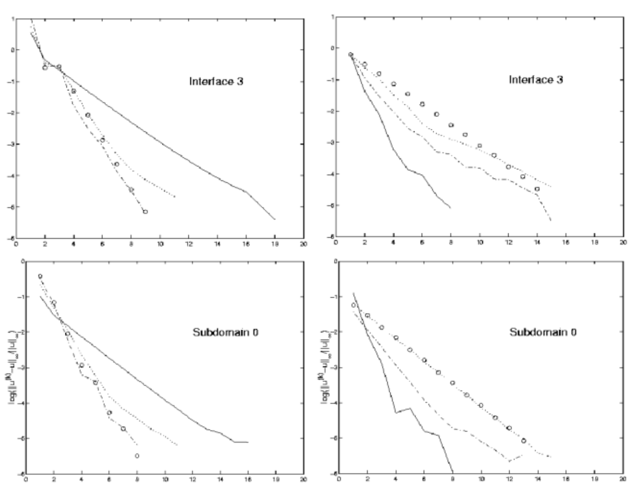
$$\Delta u + \left(1 + \tfrac{u}{1000}\right)\frac{\partial^2 u}{\partial y^2} + \left(\tfrac{1}{500}\frac{\partial u}{\partial x} + 3\right)\frac{\partial u}{\partial y} + 0.3u = 0,\ in\ \Omega_2^{IV}$$

History of convergence for PDE4 *(ROB)*



Verification of "optimum" parameters (PDE1)

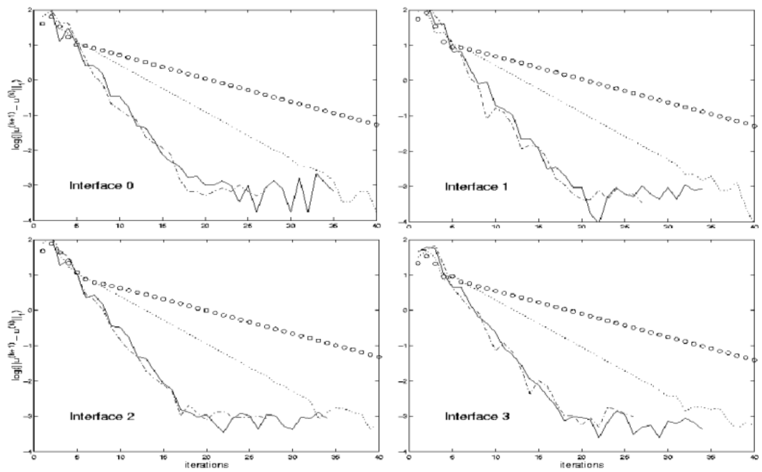
ROB



AVE

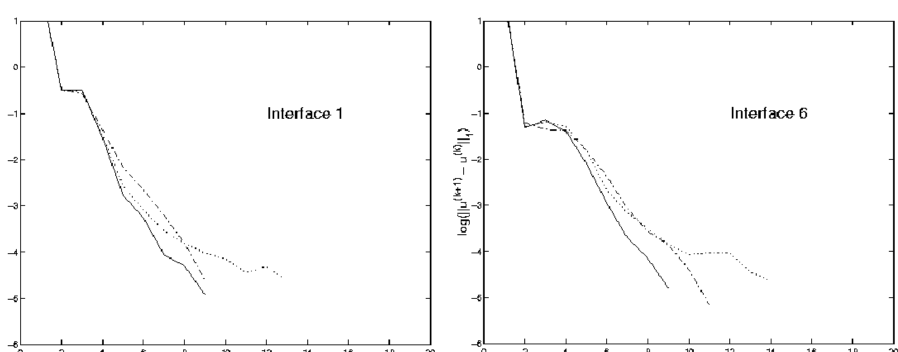
Verification of "optimum" parameters (PDE3)

ROB



Effect of descritization of the operator

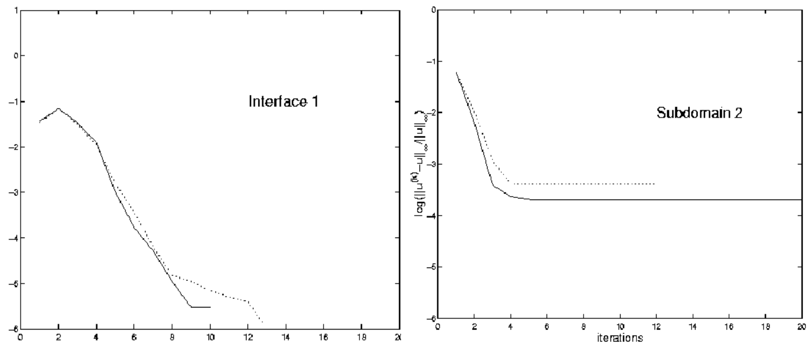
History of convergence for ***ROB*** (PDE2)



Finite Difference (5 point star)
Collocation
Bi-linear Finite Elements

Effect of discretization of the domain

History of convergence for ***ROB*** with bi-linear finite elements for PDE2.



h=0.10

h=0.20

Conclusions

- The IR methodology is suitable for composed PDE problems.
- There are theoretical results for convergence, with “optimum” relaxation parameters.
- Theoretical results are verified with numerical experiments (model and general problems).

References

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- **Interface Relaxation Methods for Elliptic Differential Equations,** J.R. Rice, P. Tsompanopoulou and E.A. Vavalis, *Applied Numerical Mathematics*, **32** (2000), pp 219-245.

Thanks!

Possible project acronyms

- ACAdEMy Advanced mathematiCAl Engineering Medical
- ACCEPt AdvanCed mathematiCal Engineering Problems
- ADVISE ADVanced mathematIcal Software Engineering
- AMALTHEA Advanced MAthematicaL meTHods EnviromentAl
- AMETHyST Advanced Mathematical mETHods SofTware
- ANiMATED AdvaNced MAThematical Engineering meDical
- ANiMATE AdvaNced MAthematical meThods Engineering
- ANAEMIA AdvaNced mAthematical Engineering MedIcAl
- ANEMONE AdvaNcEd Mathematical envirOmeNtal Engineering
- ACHIEVE AdvanCed matHematIcal EnViromental Engineering
- CoMMENd advanCed Mathematical Methods ENgineering
- CoMPLETE advanCed Mathematical PLatform EnviromenTal Engineering