

Runge-Kutta and Hermite Collocation for a biological invasion problem modeled by a generalized Fisher equation

IE Athanasakis , EP Papadopoulou and YG Saridakis

Applied Math & Computers Lab, Technical University of Crete, 73100 Chania, Crete, Greece

E-mail: yiannis@science.tuc.gr

Abstract. Fisher's equation has been widely used to model the biological invasion of single-species communities in homogeneous one dimensional habitats. In this study we develop high order numerical methods to accurately capture the spatiotemporal dynamics of the generalized Fisher equation, a nonlinear reaction-diffusion equation characterized by density dependent non-linear diffusion. Working towards this direction we consider strong stability preserving Runge-Kutta (RK) temporal discretization schemes coupled with the Hermite cubic Collocation (HC) spatial discretization method. We investigate their convergence and stability properties to reveal efficient HC-RK pairs for the numerical treatment of the generalized Fisher equation. The Hadamard product is used to characterize the collocation discretized non linear equation terms as a first step for the treatment of generalized systems of relevant equations. Numerical experimentation is included to demonstrate the performance of the methods.

1. Introduction

One way to incorporate density-dependent species motility, such as small scale migrations, to the Fisher-Kolmogorov ([1], [2]) classical biological invasion model, is by replacing the constant diffusion coefficient D by a density-dependent $D(u)$ one (cf. [3], [4], [5] and the references therein). Assuming that the diffusivity depends linearly on density (cf. [6]), namely $D(u) = \lambda_0 u + \lambda_1$, the generalized Fisher's equation takes the form:

$$u_t = [(\lambda_0 u + \lambda_1)u_x]_x + \lambda_2 u - \lambda_3 u^2 \quad (1)$$

where $u \equiv u(x, t)$, $x \in [a, b]$, $t \in [0, T]$ and $\lambda_i \in \mathbf{R}$, for $i = 0, 1, 2, 3$, with $\lambda_2 \lambda_3 > 0$, while Neumann boundary conditions $u_x(a, t) = 0$ and $u_x(b, t) = 0$ are imposed at the boundaries and an initial density distribution $u(x, 0) = f(x)$ is assumed. As exact solutions have been recently derived (cf. [6]), equation (1) can play the role of a model problem for investigating the behavior of numerical techniques.

Working towards the development of high order numerical schemes, in Section 2 we implement the Hermite Collocation (HC) method to discretize in space and we use the Hadamard product to characterize the non linear equation terms. In Section 3, explicit (to avoid solving nonlinear systems) Strong Stability Preserving (SSP) Runge-Kutta time discretization schemes are coupled with the HC method. Their efficiency, convergence and ability to maintain strong stability are investigated in Section 4 through numerical experimentation.



2. Hermite Collocation (HC) Spatial Discretization Method

Assuming sufficiently smooth solutions of equation (1), a uniform partition of $[a, b]$ into N subintervals, with spacing $h = (b - a)/N$ and nodes $x_j := a + jh$, $j = 1, \dots, N + 1$, the Hermite Collocation method seeks $O(h^4)$ approximations in the form:

$$U(x, t) = \sum_{j=1}^{N+1} [\alpha_{2j-1}(t)\phi_{2j-1}(x) + \alpha_{2j}(t)\phi_{2j}(x)] \quad (2)$$

where $\phi_{2j-1}(x)$ and $\phi_{2j}(x)$ are the Hermite cubic basis functions, defined by

$$\phi_{2j-1}(x) = \begin{cases} \phi\left(\frac{x_j-x}{h}\right) & , \quad x \in I_{j-1} \\ \phi\left(\frac{x-x_j}{h}\right) & , \quad x \in I_j \\ 0 & , \quad \text{otherwise} \end{cases} \quad , \quad \phi_{2j}(x) = \begin{cases} -h\psi\left(\frac{x_j-x}{h}\right) & , \quad x \in I_{j-1} \\ h\psi\left(\frac{x-x_j}{h}\right) & , \quad x \in I_j \\ 0 & , \quad \text{otherwise} \end{cases} \quad (3)$$

with $I_j = [x_j, x_{j+1}]$, $j = 1, \dots, N$, and $\phi(s) = (1-s)^2(1+2s)$, $\psi(s) = s(1-s)^2$ for $s \in [0, 1]$.

Substitution of the approximate solution (2) into equation (1) yields a residual

$$\mathcal{R}(x, t) := U_t - [(\lambda_0 U + \lambda_1)U_x]_x + \lambda_2 U - \lambda_3 U^2 \quad (4)$$

The Collocation method at the Gauss points (cf. [7]) produces a system of algebraic equations, for the evaluation of the unknown parameters $\alpha_i \equiv \alpha_i(t)$, $i = 1, \dots, 2(n+1)$, by forcing the residual $\mathcal{R}(x, t)$ to vanish at the two Gaussian interior collocation points per subinterval. In doing so, the two elemental collocation equations that correspond to the element I_j , $j = 1, \dots, N$ may be written in the matrix form:

$$C_j^{(0)} \begin{bmatrix} \dot{\alpha}_{2j-1} \\ \dot{\alpha}_{2j} \\ \dot{\alpha}_{2j+1} \\ \dot{\alpha}_{2j+2} \end{bmatrix} = \lambda_0 \left[\left(C_j^{(1)} \begin{bmatrix} \alpha_{2j-1} \\ \alpha_{2j} \\ \alpha_{2j+1} \\ \alpha_{2j+2} \end{bmatrix} \right) \circ \left(C_j^{(1)} \begin{bmatrix} \alpha_{2j-1} \\ \alpha_{2j} \\ \alpha_{2j+1} \\ \alpha_{2j+2} \end{bmatrix} \right) + \left(C_j^{(0)} \begin{bmatrix} \alpha_{2j-1} \\ \alpha_{2j} \\ \alpha_{2j+1} \\ \alpha_{2j+2} \end{bmatrix} \right) \circ \left(C_j^{(2)} \begin{bmatrix} \alpha_{2j-1} \\ \alpha_{2j} \\ \alpha_{2j+1} \\ \alpha_{2j+2} \end{bmatrix} \right) \right] \\ + \lambda_1 C_j^{(2)} \begin{bmatrix} \alpha_{2j-1} \\ \alpha_{2j} \\ \alpha_{2j+1} \\ \alpha_{2j+2} \end{bmatrix} + \lambda_2 C_j^{(0)} \begin{bmatrix} \alpha_{2j-1} \\ \alpha_{2j} \\ \alpha_{2j+1} \\ \alpha_{2j+2} \end{bmatrix} - \lambda_3 \left(C_j^{(0)} \begin{bmatrix} \alpha_{2j-1} \\ \alpha_{2j} \\ \alpha_{2j+1} \\ \alpha_{2j+2} \end{bmatrix} \right) \circ \left(C_j^{(0)} \begin{bmatrix} \alpha_{2j-1} \\ \alpha_{2j} \\ \alpha_{2j+1} \\ \alpha_{2j+2} \end{bmatrix} \right) \quad (5)$$

where $\dot{\alpha}_i \equiv \frac{d\alpha_i}{dt}$ and

$$C_j^{(k)} = \frac{1}{h^k} \begin{bmatrix} s_1^{(k)} & h s_2^{(k)} & s_3^{(k)} & h s_4^{(k)} \\ s_3^{(k)} & h s_4^{(k)} & s_1^{(k)} & h s_2^{(k)} \end{bmatrix}$$

with

$$s_1 = \begin{cases} \frac{9+4\sqrt{3}}{18} \\ -1 \\ -2\sqrt{3} \end{cases} \quad s_2 = \begin{cases} \frac{3+\sqrt{3}}{36} \\ \frac{\sqrt{3}}{6} \\ -1 - \sqrt{3} \end{cases} \quad s_3 = \begin{cases} \frac{9-4\sqrt{3}}{18} \\ 1 \\ 2\sqrt{3} \end{cases} \quad s_4 = \begin{cases} \frac{3-\sqrt{3}}{36} & , \quad \text{for } k = 0 \\ -\frac{\sqrt{3}}{6} & , \quad \text{for } k = 1 \\ -1 + \sqrt{3} & , \quad \text{for } k = 2 \end{cases} \quad .$$

In the above equations the symbol \circ denotes the Hadamard matrix product. Apparently, the collocation discretized non-linear terms are formed as the Hadamard matrix product of the collocation discretized corresponding linear terms.

Furthermore, the additional two needed equations are obtained by forcing the approximate solution $U(x, t)$, to satisfy the boundary conditions. This directly implies that

$$\alpha_2 = 0, \quad \alpha_{2N+2} = 0 \quad \text{hence also} \quad \dot{\alpha}_2 = 0, \quad \dot{\alpha}_{2N+2} = 0$$

Careful assembling now all the interior and boundary collocation equations, described above, we obtain the ODE system:

$$\dot{\mathbf{a}} = C(\mathbf{a}) \tag{6}$$

$C(\mathbf{a}, t) := C^{(0)^{-1}} \left[\lambda_0 \left(C^{(1)} \mathbf{a} \circ C^{(1)} \mathbf{a} + C^{(0)} \mathbf{a} \circ C^{(2)} \mathbf{a} \right) + \lambda_1 C^{(2)} \mathbf{a} + \lambda_2 C^{(0)} \mathbf{a} - \lambda_3 C^{(0)} \mathbf{a} \circ C^{(0)} \mathbf{a} \right]$
 where $C^{(k)}$, $k = 0, 1, 2$ denotes the one dimensional Collocation matrix corresponding to the k -th derivative, $\mathbf{a} \equiv \mathbf{a}(t) = [\alpha_1 \ \alpha_3 \ \dots \ \alpha_{2N+1}]^T$ denotes the vector of unknowns and $\dot{\mathbf{a}} := \frac{d}{dt} \mathbf{a} = [\dot{\alpha}_1 \ \dot{\alpha}_3 \ \dots \ \dot{\alpha}_{2N+1}]^T$. We note that the linear independence of the Hermite cubic basic functions implies the non-singularity of the matrix $C^{(0)}$.

3. Strong Stability Preserving Runge-Kutta Time Discretization Schemes

High order strong stability preserving (SSP) Runge-Kutta methods were developed (cf. [8], [9], [10]) for the time discretization of the semi-discrete system of ODEs obtained from the spatial discretization of PDEs by a Total Variation Diminishing (TVD) finite difference or finite element method. The essence of the SSP class of time discretization methods lies on their ability to maintain strong stability in any given norm $\|\cdot\|$, that is to say

$$\|\mathbf{a}^{n+1}\| \leq \|\mathbf{a}^n\|, \quad \mathbf{a}^n := \mathbf{a}(t^n) \quad \text{with } t^n := n\Delta t, \quad n = 0, 1, \dots \tag{7}$$

assuming that forward Euler is strongly stable and providing suitable restrictions of the time stepping, while fully preserving the order of the error of the spatial discretization.

General m stage and order p SSP schemes are denoted by SSP(m,p) or SSPRK(m,p). In this work we consider two optimal (cf. [11], [12]) SSP(3,3) and SSP(4,3) schemes for the solution of the semi-discrete ODE system in (6). They can be written in the form:

<i>SSP(4, 3)</i>		<i>SSP(3, 3)</i>
$\mathbf{a}^{(1)} = \mathbf{a}^n + \frac{1}{2}\Delta t C(\mathbf{a}^n)$	$\mathbf{a}^{(1)} = \mathbf{a}^n + \Delta t C(\mathbf{a}^n)$	
$\mathbf{a}^{(2)} = \mathbf{a}^{(1)} + \frac{1}{2}\Delta t C(\mathbf{a}^{(1)})$	$\mathbf{a}^{(2)} = \frac{3}{4}\mathbf{a}^n + \frac{1}{4}\mathbf{a}^{(1)} + \frac{1}{4}\Delta t C(\mathbf{a}^{(1)})$	
$\mathbf{a}^{(3)} = \frac{2}{3}\mathbf{a}^n + \frac{1}{3}\mathbf{a}^{(2)} + \frac{1}{6}\Delta t C(\mathbf{a}^{(2)})$	$\mathbf{a}^{n+1} = \frac{1}{3}\mathbf{a}^n + \frac{2}{3}\mathbf{a}^{(2)} + \frac{2}{3}\Delta t C(\mathbf{a}^{(2)})$	
$\mathbf{a}^{n+1} = \mathbf{a}^{(3)} + \frac{1}{2}\Delta t C(\mathbf{a}^{(3)})$		

We note that the classical RK4 is also considered and used in our experiments for comparison purposes.

4. Numerical Results

The model problem used to investigate the performance of the HC-RK methods is given by

$$u_t = [(1-u)u_x]_x + 2u - 2u^2, \quad -5\pi/2 \leq x \leq 5\pi/2, \quad 0 \leq t \leq T$$

$$u_x\left(\frac{-5\pi}{2}, t\right) = 0, \quad u_x\left(\frac{5\pi}{2}, t\right) = 0, \quad u(x, 0) = \frac{1}{3} [2 + \sin(-x)] \tag{8}$$

and admits the analytical solution $u(x, t) = \frac{1}{3} \left[\frac{e^{-t}(3e^{2t} + 1 + 2\sin(-x))}{e^t + e^{-t}} \right]$. The spatial absolute error used in all experiments is defined as $\mathcal{E}^n := \|U(x, t^n) - u(x, t^n)\|_2$ and the necessary time stepping restrictions imposed are $\Delta t \leq \frac{1}{5}h^2$ for SSP(4,3), $\Delta t \leq \frac{1}{10}h^2$ for SSP(3,3) and $\Delta t \leq 0.3h^3$ for RK4. Under these restrictions all time discretization schemes were strongly stable, as it is depicted in Figure 1b for SSP(4,3) while, at the same time, the $O(h^4)$ order of convergence of the HC method is preserved (see Table I). The maximum $\mathcal{E}^1 = \max_n \{\mathcal{E}^n\}$ error norm and the computational time needed to reach the time level $t = 1$ are also included in Table I to demonstrate the efficiency of the methods.

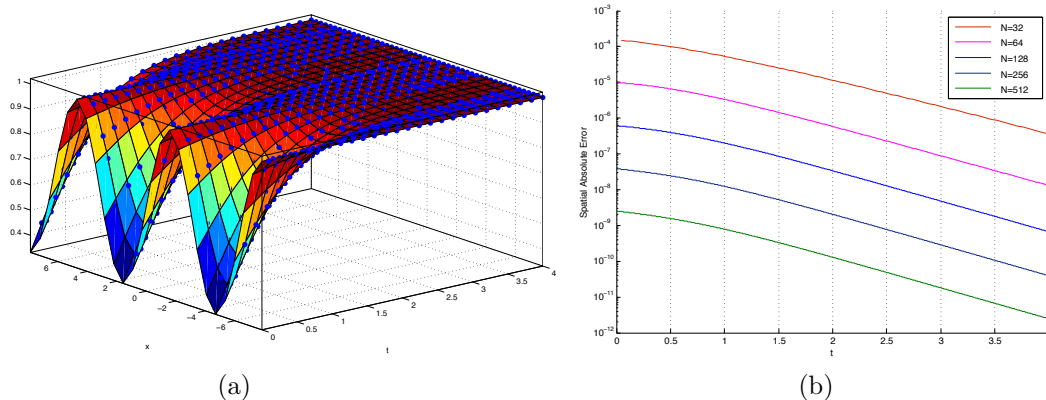


Figure 1. a) Exact (solid) and HC-SSPRK(4,3) approximate (point) solutions for $N = 64$
 b) Spatial absolute error as a function of time for the HC-SSPRK(4,3)

TABLE I Computational Performance of HC-RK schemes

N	Spatial Absolute Error at $t = \Delta t$			Spatial Order of Convergence			Time (sec)		
	SSP(4,3)	SSP(3,3)	RK4	SSP(4,3)	SSP(3,3)	RK4	SSP(4,3)	SSP(3,3)	RK4
32	1.53e-04	1.56e-04	1.53e-04	-	-	-	0.01	0.03	0.02
64	9.85e-06	9.89e-06	9.89e-06	3.95	3.98	3.95	0.05	0.05	0.15
128	6.20e-07	6.20e-07	6.21e-07	3.99	3.99	3.99	0.14	0.20	1.60
256	3.88e-08	3.89e-08	3.89e-08	4.00	4.00	4.00	0.72	1.07	19.78
512	2.56e-09	2.56e-09	2.56e-09	3.93	3.93	3.93	4.28	6.48	269.91

5. Conclusion

In this work, the fourth order *HC* is coupled to third order *SSPRK* schemes for the treatment of a generalized Fisher equation. Numerical evidence is presented for the method's efficient and stable behaviour (Table I) as well as the method's ability to maintain strong stability (Fig. 1b).

Acknowledgement

The present research work has been co-financed by the European Union (European Social Fund ESF) and Greek national funds through the Operational Program Education and Lifelong Learning of the National Strategic Reference Framework (NSRF) - Research Funding Program: THALIS. Investing in knowledge society through the European Social Fund.

References

- [1] Fisher RA *The wave of advance of advantageous genes*, Ann. Eugen., **7**, 255-369, 1937
- [2] Kolmogorov AN, Petrovsky IG and Piskunov NS *Investigation of the equation of diffusion combined with increasing of the substance and its application to a biology problem*, Bull. Moscow State Univ. Ser. A: Math. and Mech., **1(6)**, 1-25, 1937
- [3] Hengeveld R *Dynamics of Biological Invasions*, Chapman and Hall, London, 1989
- [4] Murray JD *Mathematical Biology*, Springer, Berlin, 1989
- [5] Petrovskii SV and Li BL *Exactly Solvable Models of Biological Invasion*, Taylor & Francis, 2010
- [6] Cherniha R and Dutka V *Exact and Numerical Solutions of the Generalized Fisher Equation*, Reports on Mathematical Physics, **47**, 393-412, 2001
- [7] de Boor C and Swartz B *Collocation at Gaussian points*, SIAM Num. Anal., vol. 10, pp. 582-606, 1973
- [8] Shu CW *Total-variation-diminishing time discretizations*, SIAM J. Sci. Stat. Comput., **9**, 1073-1084, 1988
- [9] Shu CW and Osher S *Efficient implementation of essentially non-oscillatory shock-capturing schemes*, J. Comput. Phys., **77**, 439-471, 1988
- [10] Gottlieb S, Shu CW and Tadmor E *Strong Stability-Preserving High-Order Time Discretization Methods*, SIAM Num. Anal., **43**, 89-112, 2001
- [11] Gottlieb S and Shu CW *Total variation diminishing Runge-Kutta schemes*, Mat. Comp., **67**, 73-85, 1998
- [12] Ruuth S and Spiteri R *Two barriers on strong-stability-preserving time discretization methods*, J. Scientific Computation, **17**, 211-220, 2002